Motu Working Paper 26-02

On quantitative and graphical measures of the severity of New Zealand's recessions and strength of its expansions

economic & public policy research

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February 2026

Document information

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Acknowledgements

We appreciate valuable comments from a presentation to VUW's School of Economics & Finance on 23 October 2024.

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 ISSN 1176-2667 (Print), ISSN 1177-9047 (Online).

Abstract

We develop three measures for the shape of business cycle phases, reflecting excess gains and losses relative to constant quarterly growth across the phase. These measures can be seen as representing better or worse economic outcomes during recession and expansion phases relative to constant growth rate paths and, as a consequence, provide summary evaluation measures for such economic outcomes. Using a phase's constant quarterly growth rate as the benchmark, our methodology builds on Harding and Pagan (2016) by developing quantitative measures that have useful economic interpretations and which are amenable to informative graphical display and analysis. Empirical outcomes are provided for New Zealand's recession and expansion phases.

JEL codes

C1, C22, C82, E32

Keywords

Business cycle phase shape measurement; excess gain and loss measures; business cycle turning points; business cycle recessions and contractions; New Zealand

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1 Introduction

Traditionally, assessment measures for business cycles and phases have been confined to *phase duration* and *amplitude*, where sometimes either or both of these have been referred to as severe. For example, equating severity of a recession with depth or amplitude of its phase can be traced back at least as far as Bry and Boschan (1971, ch. 4).

More recently, Harding and Pagan (2016, ch. 5.5) provided a valuable summary of measures of the shape of the paths of business cycle phases in addition to duration and amplitude. These measures build on the concepts of cumulated loss or gain relative to no growth of output, and of *excess area* relative to a constant rate of growth.[1] These concepts are consistent with the view that the *severity* of a recession and the *strength* of an expansion should ideally reflect the shape of their respective paths across the phase as well as their duration and amplitude.

For New Zealand's post-Second World War recessions and expansion paths up to 2014q3, Hall and McDermott (2016) have presented empirical estimates of cumulated gains and losses relative to no growth of output. They did not, however, investigate the usefulness of excess cumulated gains and losses relative to quarterly constant growth rate paths.

This paper adds to our understanding of New Zealand's post-Second World War business cycles in two key ways:

We develop three measures for the shape of business cycle phases, reflecting excess gains and losses relative to constant quarterly growth across the phase (rather than no growth). These measures can be seen as representing better or worse economic outcomes during recession and expansion phases relative to constant growth rate paths and, as a consequence, provide summary evaluation measures for such economic outcomes. Using a phase's constant quarterly growth rate as the benchmark, our methodology builds on Harding and Pagan (2016) by developing quantitive measures that have useful economic interpretations and which are amenable to informative graphical display and analysis.

We update the Hall and McDermott (2016) classical real GDP turning-points to cover the period 2014q4 to 2025q3, thereby additionally covering New Zealand's Covid-19 recession and expansion phases, and its recent period of little to no growth. This enables us to provide empirical estimates for excess gains and losses of $\log GDP$, relative to their constant quarterly growth rates, and hence for the severity of New Zealand's recessions and the strength of its expansions.

Section 2 provides the methodology for the measures of shape of our business cycle phases, empirical results are presented in Section 3 and conclusions are given in Section 4.

[1] Harding and Pagan also offer measures of asymmetry, steepness and deepness, sharpness, diversity of phases, plucking effects and recovery times, and duration dependence in phases. Consideration of such measures is beyond the scope of this paper.

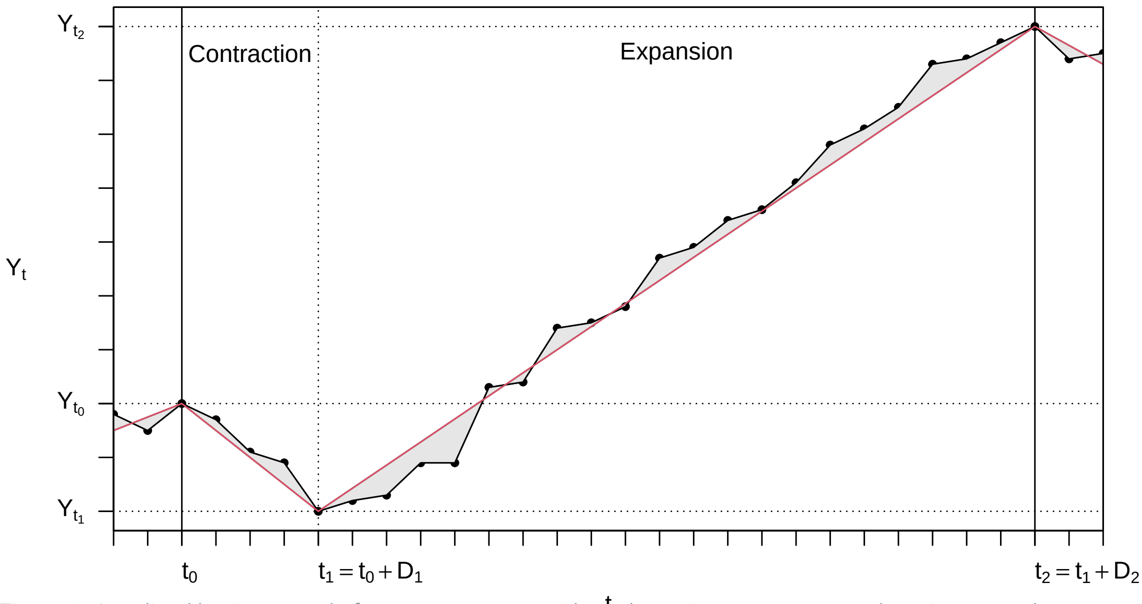

Figure 1. A stylised business cycle for $Y_t = \log GDP$ with peaks at times t_0, t_2, a trough at time t_1 and a contraction phase of $D_1 = 4$ quarters followed by an expansion phase of $D_2 = 21$ quarters. The piecewise-linear trend $\hat{Y}_t$ (red) is superimposed.

2 Methodology

Let $Y_t = \log y_t$ denote the logarithms of quarterly real New Zealand gross domestic product y_t and assume that Y_t has been segmented into alternating phases of contraction (C) and expansion (E) where each consecutive pair of phases (CE or EC) provides a realisation of the business cycle. Denote the *phase onsets* by the turning points t_j ($j = 0, \ldots, p$) where there are p complete phases and the t_j have been determined by a suitable segmentation algorithm such as the BB algorithm (Bry and Boschan, 1971) or the BBQ method (Harding and Pagan, 2002, 2016) among other such methods. A stylised plot of a CE business cycle and its constituent phases is given in Figure 1.

The piecewise-linear trend $\hat{Y}_t$ formed by linear interpolation between the turning points provides a good first approximation to Y_t as exemplified in Figure 1. Within each phase $\hat{Y}_t$ is given by

$$\hat{Y}_{t_{j-1}+k} = Y_{t_{j-1}} + k\hat{\mu}_j \qquad (j = 1, \ldots, p,\ k = 0, \ldots, D_j) \tag{1}$$

where $D_j = t_j - t_{j-1}$ is the *duration* of phase j, the slope $\hat{\mu}_j$ is given by

$$\hat{\mu}_j = \frac{Y_{t_j} - Y_{t_{j-1}}}{D_j} = \frac{A_j}{D_j}$$

and A_j is the *amplitude* of phase j. Note that $\hat{Y}_{t_{j-1}} = Y_{t_{j-1}}$ and $\hat{Y}_{t_{j-1}+D_j} = Y_{t_j}$.

The parameters D_j, A_j and $\hat{\mu}_j$ are key summary measures describing the basic properties of the path or trajectory of Y_t within phase j. The duration D_j is measured in quarters and, since differences between values of Y_t are close approximations to proportionate differences in the corresponding values of y_t (provided these are not large), the amplitude A_j can be interpreted as the proportionate change or *growth* in GDP over phase j. Furthermore

$$\hat{\mu}_j = \frac{Y_{t_j} - Y_{t_{j-1}}}{D_j} = \frac{1}{D_j}\sum_{k=1}^{D_j}(Y_{t_{j-1}+k} - Y_{t_{j-1}+k-1}) \tag{2}$$

can be interpreted as the average quarterly *growth rate* in phase j.

Given suitable turning points (t_j, Y_{t_j}), the piecewise-linear trend $\hat{Y}_t$ is completely determined. It has the shortest path between turning points with constant growth rate $\hat{\mu}_j$ in each quarter of phase j. However more can be said about the provenance of the piecewise-linear trend. Since many economic time series such as Y_t are non-stationary and well-modelled by integrated processes (typically $I(1)$), a simple model for the growth rates of Y_t in phase j is

$$\Delta Y_t = Y_t - Y_{t-1} = \mu_j + \epsilon_t \qquad (t = t_{j-1}+1, \ldots, t_{j-1}+D_j) \tag{3}$$

which is a random walk with drift μ_j. Here the innovations ϵ_t constitute a zero-mean stationary process, Δ is the difference operator, and Y_t is an $I(1)$ process with phase-dependent mean. For this simple model the OLS estimator of μ_j is identical to $\hat{\mu}_j$ given by (2). In particular, when integrated over the quarters within phase j, the fitted constant growth rate model yields the piecewise-linear trend (1) for Y_t.

For these reasons, *we adopt the piecewise-linear trend (constant quarterly growth within phases) as our benchmark* with which to evaluate suitable descriptive measures of each individual phase and, subsequently, provide comparisons across phases. This differs from the measures proposed by Harding and Pagan (2002, 2016) which use no growth within phases as their benchmark. For ease of exposition, we now suppress reference to the phase index j in the sections that follow.

2.1 Phase analysis

Consider the deviations between Y_t and the benchmark piecewise-linear trend $\hat{Y}_t$. The average trend deviation over any given phase is given by

$$\bar{E}_* = \frac{E_*}{D}, \qquad E_* = \sum_{k=1}^{D}(Y_{t_*+k} - \hat{Y}_{t_*+k}) \tag{4}$$

where t_* denotes the phase onset time, E_* is the sum and $\bar{E}_*$ the mean of the trend deviations across the phase. To a good approximation, $\bar{E}_*$ can be interpreted as the mean proportionate difference between the levels y_t and $\hat{y}_t = \exp \hat{Y}_t$, the corresponding trend in levels. In particular, for any given phase,

$$\bar{E}_* \approx \frac{1}{D}\sum_{k=1}^{D}\frac{y_{t_*+k}e^{-k\hat{\mu}} - y_{t_*}}{y_{t_*}}, \qquad E_* \approx \frac{\sum_{k=1}^{D}(y_{t_*+k}e^{-k\hat{\mu}} - y_{t_*})}{y_{t_*}}, \tag{5}$$

so that $\bar{E}_*$ can be interpreted as the mean proportionate difference across the phase between quarterly $GDP\ y_{t_*+k}$ and initial $GDP\ y_{t_*}$, with the former adjusted for constant quarterly growth over the phase. Furthermore E_* can be interpreted as a net present value of accumulated GDP flows (gains or losses) over the phase expressed as a price relative to the value of GDP at phase onset. The discount (hurdle) rate is the constant growth rate within phases.

Within any given phase the trend deviations $Y_t - \hat{Y}_t$ measure the excess $\log GDP$ over constant quarterly growth. This is positive when constant quarterly growth is exceeded and negative otherwise. As a consequence E_* provides a measure of the *total excess* $\log GDP$ *over constant quarterly growth* and $\bar{E}_*$ gives the *mean quarterly excess over constant quarterly growth* for each phase.

If the path of Y_t is close to constant growth within the given phase then $\bar{E}_*$ and E_* will necessarily be close to zero. However the converse, although often the case, is not always true. Since Y_t is strongly autocorrelated, the path of Y_t over any phase will often exhibit relatively smooth deviations away from the piecewise-linear trend. In such cases, values of $\bar{E}_*$ close to zero will often imply that Y_t is close to the piecewise-linear trend and so will generally support the hypothesis of a constant growth rate over the phase.

Following in the footsteps of Harding and Pagan (2016) we note that, for each phase, the total excess E_* exactly calculates the area between the path of Y_t and that of constant growth with the proviso that areas above the constant growth trend (positive excess) are positive and those below the trend (negative excess or deficit) are negative. This is illustrated by the shaded area in Figure 1 and follows since E_* is the sum of quarterly trapezoidal areas[2]. This observation is particularly important for graphical analysis and made more useful by the smoothness of Y_t and its deviations from constant growth across the phase. Furthermore, since the values of Y_t within any phase will generally be bounded by Y_{t_*} and Y_{t_*+D} (a consequence of the procedure used to determine turning points), the value of E_* will have maximum area $\frac{1}{2}D|A|$ above and $-\frac{1}{2}D|A|$ below the constant growth path within phases, where $|A|$ is the absolute value of the amplitude A. This implies that $\bar{E}_*$ is also bounded with

$$-\tfrac{1}{2}D|A| < E_* < \tfrac{1}{2}D|A|, \qquad -\tfrac{1}{2}|A| < \bar{E}_* < \tfrac{1}{2}|A| \tag{6}$$

in practice (noisy series may create exceptions in some cases). As noted in Harding and Pagan (2016), the maximum area $\frac{1}{2}D|A|$ provides a good measure (graphically as well as numerically) of the absolute magnitude of any given phase in terms of its impact on GDP.

If the path of Y_t tracks significantly above the piece-wise linear trend in a given phase (an early strong recovery followed by weak growth in the case of expansions, or a weak decline followed by a late sharp decline in the case of contractions) then E_* and $\bar{E}_*$ will tend to be large positive and close to their upper bounds. Similar comments apply when Y_t tracks significantly below the piece-wise linear trend (weak growth followed by a late strong recovery in the case of expansions, or an early sharp decline followed a weak decline in the case of contractions). In the latter case E_* and $\bar{E}_*$ will be large negative and close to their lower bounds. In general, large positive values of E_*, $\bar{E}_*$ will tend to be associated with a stronger economic performance and large negative values with a weaker economic performance over the phase.

However E_* has the deficiency that its value is dependent on both the phase amplitude A and duration D, while the value of $\bar{E}_*$ is dependent on the amplitude A (increases in either A or D will typically lead to increases in E_*, whereas increases in A will typically lead to increases in $\bar{E}_*$). While both measures have independent utility, these dependencies make it difficult to compare their values across phases unless they have similar durations (E_*) and amplitudes (E_* and $\bar{E}_*$).

To correct for this dependence we introduce the *standardised quarterly excess over constant quarterly growth*

$$\tilde{E}_* = \frac{E_*}{D|A|} = \frac{\bar{E}_*}{|A|} \tag{7}$$

which evaluates $\bar{E}_*$ across the phase using the scaled values $(Y_{t_*+k} - Y_{t_*})/|A|$ and $(\hat{Y}_{t_*+k} - Y_{t_*})/|A|$. The latter is just the time from onset expressed as a proportion of the phase duration D multiplied by the sign of $\hat{\mu}$, whereas $(Y_{t_*+k} - Y_{t_*})/|A|$ measures the increase of Y_{t_*+k} from Y_{t_*} as a proportion of the absolute phase amplitude $|A|$ with this interpretation also holding, to a good approximation, for the levels y_t. In practice, from (6), -1/2 $< \tilde{E}_* <$ 1/2 provided the phase path of Y_t is sufficiently smooth.

The standardised excess $\tilde{E}_*$ provides a useful measure of the shape of the path of Y_t across the phase

[2] The area of a trapezoid is the average length of its two parallel sides multiplied by their distance apart.

Table 1. Duration D, amplitude A, growth rate $\hat{\mu}$ and the total excess E_*, mean quarterly excess $\bar{E}_*$, standardised quarterly excess $\tilde{E}_*$ over constant quarterly growth for 6 phases of New Zealand $\log GDP$ over the period 1990q4 to 2019q4. All figures are percentages except for duration D which is in quarters.

Contraction							Expansion						
Onset	D	A	$\hat{\mu}$	E_*	$\bar{E}_*$	$\tilde{E}_*$	Onset	D	A	$\hat{\mu}$	E_*	$\bar{E}_*$	$\tilde{E}_*$
1990q4	2	-3.13	-1.56	-0.89	-0.44	-14.16	1991q2	24	23.54	0.98	-23.42	-0.98	-4.15
1997q2	3	-0.94	-0.31	0.30	0.10	10.64	1998q1	39	35.51	0.91	24.51	0.63	1.77
2007q4	6	-2.55	-0.43	0.33	0.06	2.17	2009q2	42	30.40	0.72	-59.74	-1.42	-4.68

which can be used to *compare the shape of the path of Y_t across phases of the same type (C or E), but different durations and amplitudes*. It measures the mean quarterly excess over constant quarterly growth expressed as a proportion of absolute amplitude $|A|$. From (5), $\tilde{E}_*$ is well-approximated in terms of levels by

$$\tilde{E}_* \approx \frac{1}{D}\sum_{k=1}^{D}\frac{y_{t_*+k}e^{-k\hat{\mu}} - y_{t_*}}{|y_{t_*+D} - y_{t_*}|},$$

so that the standardised excess $\tilde{E}_*$ can be interpreted as the mean of discounted quarterly GDP flows (gains or losses over value at onset) expressed as a proportion of absolute GDP change ($|y_{t_*+D} - y_*|$) over the phase. As before, the discount rate is the constant growth rate $\hat{\mu}$ across the phase. These considerations provide further support for the use of $\tilde{E}_*$ as a measure for comparing the shapes of phase paths across phases, independent of their duration and amplitude.

Furthermore, it can be shown that

$$\tilde{E}_* = \frac{1}{D}\sum_{k=1}^{D}(1 - \frac{k}{D})\frac{\Delta Y_{t_*+k} - \hat{\mu}}{|\hat{\mu}|} \tag{8}$$

so that, for each phase, $\tilde{E}_*$ is just the sample covariance of the standardised quarterly growth rates $(\Delta Y_{t_*+k} - \hat{\mu})/|\hat{\mu}|$ with the proportion of time remaining in the phase $(1-k/D)$. Apart from a constant of proportionality, the measure $\tilde{E}_*$ is essentially a proxy for a correlation since it is the sample covariance of dimensionless quantities.

Examples of the excess measures for 6 consecutive phases of New Zealand $\log GDP$ over the period 1990q4 to 2019q4 are illustrated in Table 1 and Figure 2 with a more comprehensive analysis provided in Section 3. Table 1 gives the values of duration D, amplitude A, growth rate $\hat{\mu}$ and the total excess E_*, mean quarterly excess $\bar{E}_*$, standardised quarterly excess $\tilde{E}_*$ over constant quarterly growth for each phase. In all cases the three excess measures agree in their rankings (high values score better than low), although this will not always be the case. Overall, the 1997q2 contraction phase and 1998q1 expansion phase performed best while the 1990q4 contraction phase and 2009q2 expansion phase performed worst.

The various differences in value and interpretation of the three excess measures can be deduced from the phase plots given in Figure 2. Here the top panels plot the path of $Y_{t_*+k} - Y_{t_*}$ against time from onset (quarters), the middle panels against time from onset (percent duration), and the bottom panels plot the standardised path $(Y_{t_*+k} - Y_{t_*})/|A|$ against time from onset (both expressed as percentages). The corresponding constant growth rate trends $\hat{Y}_{t_*+k} - Y_{t_*}$ (middle and top panels) and $(\hat{Y}_{t_*+k} - Y_{t_*})/|A|$ (bottom panels) are also superimposed with the shaded areas depicting E_* (top panels), $\bar{E}_*$ (middle plots) and $\tilde{E}_*$ (bottom panels). In general and as might be expected, the phase paths become more comparable as we progress from the top plots (total excesses), to the middle plots (mean excesses) and bottom plots (standardised excesses).

In addition to their respective excess measures and phase paths, the top panels show the differences

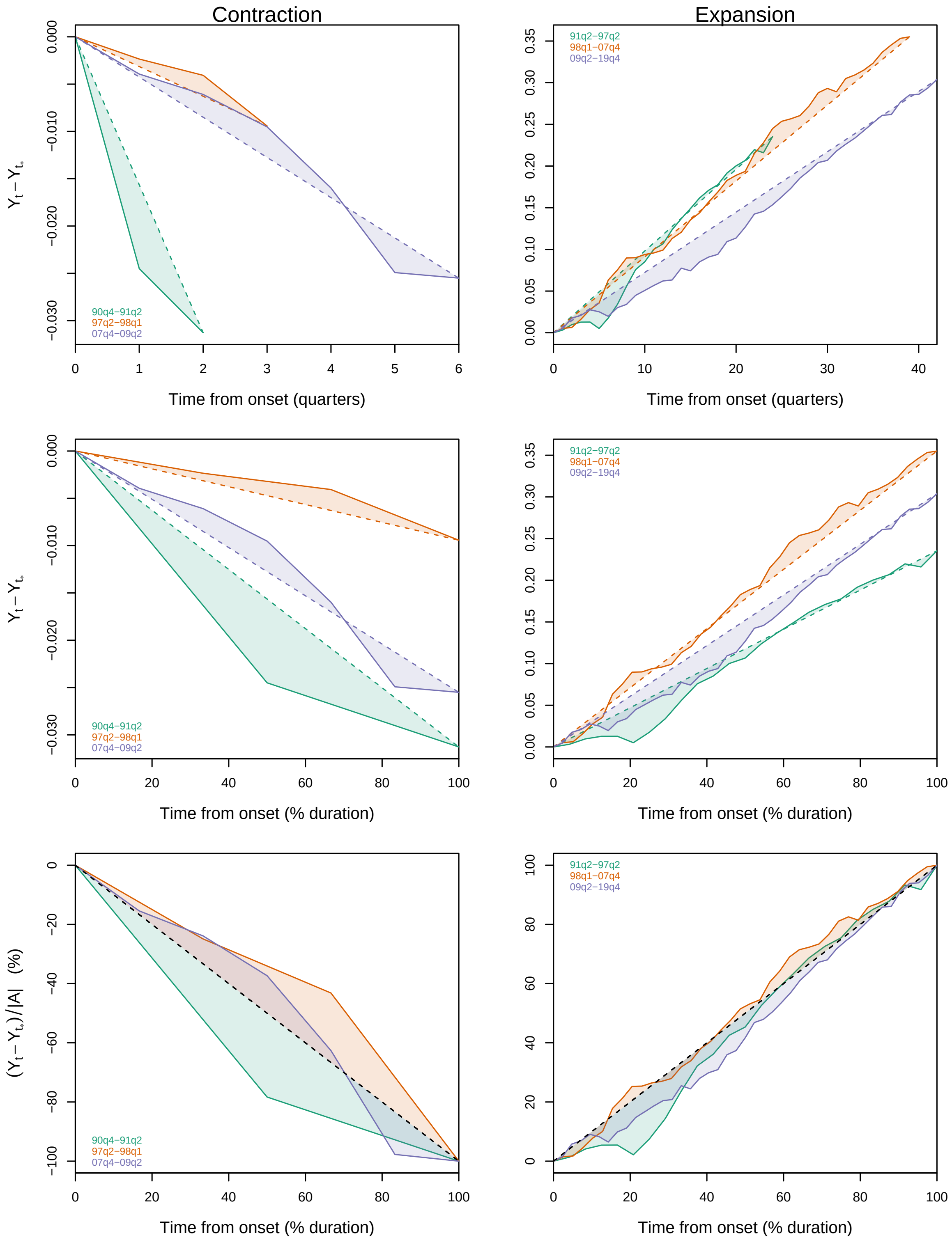

Figure 2. Contraction and expansion phase plots for the 6 phases of New Zealand $\log GDP$ *over the period 1990q4 to 2019q4. The top 4 plots show the paths of* $Y_{t_*+k} - Y_{t_*}$ *and their constant growth rate trends* $\hat{Y}_{t_*+k} - Y_{t_*}$ *against time from onset (quarters for top plots; percent duration for middle plots). The bottom plots show the standardised paths* $(Y_{t_*+k} - Y_{t_*})/|A|$ *and their constant growth rate trends* $(\hat{Y}_{t_*+k} - Y_{t_*})/|A|$ *against time from onset (percent duration). The shaded areas depict* E_* *(top plots),* $\bar{E}_*$ *(middle plots) and* $\tilde{E}_*$ *(bottom plots).*

in duration D, amplitude A and quarterly growth rate $\hat{\mu}$ between the phases with the middle panels showing the differences in amplitude A. The bottom panels show just the phase paths and their shape measures $\tilde{E}_*$. Notably, the top panels provide an informative multidimensional graphical summary of the original phase paths under study and their key parameters, D, A, $\hat{\mu}$ and total excess E_*. They can be used to informally identify phases with broadly similar properties as well anomalous phases that may warrant further examination and analysis.[3]

Figure 2 shows that relative economic performance, as measured by the three excess measures (shaded areas), varies across the phases. For the contraction phases, the worst performing 1990q4 phase has total excess (a loss) of almost three times the magnitude of the best performing 1997q2 phase, yet the standardised excess rates the magnitudes of these two phases as much the same. Similarly, the 1997q2 and 2007q4 contraction phases have very similar total excesses, but markedly different standardised excesses. Standardisation evidently makes a difference. For the expansion phases, the total excesses are well-differentiated with the 1991q2 and 1998q1 phases having much the same magnitudes, but differing signs, while the total excess of the worst performing 2009q2 phase (a loss) has over twice the magnitude of the best performing 1998q1 phase. However the mean and standardised excesses rate the 1991q2 and 2009q2 phases as roughly the same.

We conclude this section with a brief discussion of the *cumulated gain or loss* measure F and the *excess area* measure E proposed by Harding and Pagan (2016) which use no growth as their benchmark, rather than constant quarterly growth adopted in this paper. These measures are given by

$$F = \sum_{k=1}^{D} (Y_{t_*+k} - Y_{t_*}) - \tfrac{1}{2}A, \qquad E = \frac{F - \frac{1}{2}DA}{\frac{1}{2}D|A|}. \tag{9}$$

Here F is the total accumulation of the quarterly gains/losses in $\log GDP$ over the phase, relative to the value of $\log GDP$ at phase onset, adjusted by the correction term $\frac{1}{2}A$. The purpose of the latter is to ensure that F equals the area below the phase path Y_{t_*+k} and above Y_{t_*}. As noted in Harding and Pagan (2016) and as illustrated in Figure 1, this area is often well-approximated by the area of the right-angle triangle with base D quarters, magnitude $|A|$ and hypotenuse with slope $\hat{\mu}$, the constant growth rate over the phase. Standardising F by this area ($\frac{1}{2}D|A|$) yields the excess area measure E.

In terms of the levels y_t, F is well-approximated by

$$F \approx \frac{\sum_{k=1}^{D}(y_{t_*+k} - y_{t_*}) - \frac{1}{2}(y_{t_*+D} - y_{t_*})}{y_{t_*}}$$

which is the total accumulation of quarterly gains/losses in GDP over the phase, adjusted for the levels amplitude and expressed as a price relative to the value of GDP at phase onset. As before, the amplitude correction ensures that the numerator is just the area below the levels phase path y_{t_*+k} and above y_{t_*}. In terms of the measures proposed here

$$F = \tfrac{1}{2}DA + E_*, \qquad E = 2\tilde{E}_* \tag{10}$$

with the magnitude of F typically dominated by $\frac{1}{2}DA$ and E_* an accumulated error term. *Moreover the phase shape measures E and $\tilde{E}_*$ are identical apart from a constant factor.* It is also noted that F can be written as

$$F = \sum_{k=1}^{D} (\tfrac{1}{2}(Y_{t_*+k} + Y_{t_*+k-1}) - Y_{t_*})$$

which is just the total accumulation of the quarterly gains/losses of a smoothed version of $\log GDP$

[3] More formal clustering methods involving suitable multivariate criteria could also be used for grouping phases, but this has not been considered here.

across the phase (a 2 point uniform moving average) relative to the value of $\log GDP$ at phase onset. While F measures a total accumulated cost per phase relative to no growth, in some cases the mean cost per quarter F/D may have greater utility. Measuring the latter as a proportion of absolute amplitude leads to E and $\tilde{E}_*$.

While the phase plots in Figure 2 are relatively straightforward to visualise and compare, this may not always be the case for more noisy phase paths or where we wish to compare a larger number of phases. In such cases the phase paths can be lightly smoothed to remove visually distracting noise at little cost to the accuracy of the excess measures. Further details are given in the Appendix.

3 Empirical results

We present empirical results for New Zealand's contraction and expansion phases from 1954q2 to 2025q3, utilising an updated version of Hall and McDermott's (2011, 2016) quarterly seasonally adjusted real GDP series. Although this quarterly $GDPP$ series is available from 1947q2, Hall and McDermott (2011, pp 10, 12, 18) have commented that data for the period 1947q2 to 1954q1 are potentially of the lowest quality, through having been generated from temporally disaggregated unofficial annual estimates. This period also suffers from having reflected some particularly volatile fluctuations, associated with the conversion of New Zealand's economic production from wartime to peacetime mode.

Our preferred underlying business cycle turning points and their associated traditionally-reported key measurement characteristics have been derived using Bry-Boschan's (1971) original business cycle dating methodology (BB). These outcomes are reported in Section 3.1 and have been confirmed by alternative dating procedures based on variants of both BB and BBQ, the quarterly business cycle dating procedure proposed by Harding and Pagan (2002). In the latter case the New Zealand data needed to be prefiltered using a light smooth, an HP filter or sharpened HP filter (Hall and Thomson, 2024) with cut-off period 6 quarters.[4]

Section 3.1 provides an overview of New Zealand's real GDP cycles from 1954q2 to 2025q3. Sections 3.2 and 3.3 give our results on the severities of recessions and strengths of expansions respectively, and conclusions are drawn in Section 4. All computations and graphical analysis were carried out in the R statistical environment (R Core Team, 2024).

3.1 New Zealand real GDP business cycles.

Figure 3 shows a plot of the logarithms of quarterly, seasonally adjusted, New Zealand real GDP over the period 1954q2 to 2025q3. The contraction and expansion phases are also shown together with the piecewise linear trend illustrating the phase growth rates. Table 2 gives the turning points, durations, amplitudes and growth rates for the contraction and expansion phases of New Zealand's real GDP business cycles over the period 1954q2 to 2025q3, together with various summary statistics.

New Zealand has experienced 9 completed recessions since the peak of 1966q4, with a mean duration of 3.7 quarters. The mean phase amplitude has been a decline of 3.6 percent, and the mean growth rate per quarter has been a decline of 1.3 percent. Of the three shortest (2 quarter) contractions, one is the Covid-19 recession from 2019q4 to 2020q2, and another the most recent contraction from 2024q1 to 2024q3. These follow the second longest duration of 6 quarters for the GFC (global financial crisis)

[4] See Harding and Pagan (2016, p 31) for discussion on situations where such smoothing may be desirable.

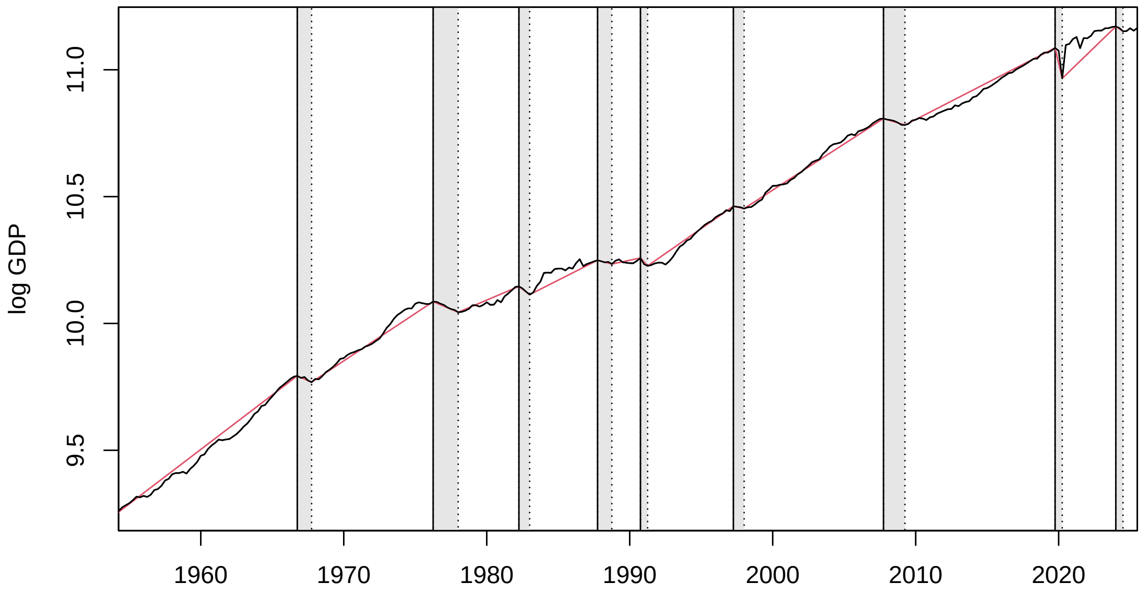

Figure 3. New Zealand classical business cycle phases 1954q2 – 2025q3. The contraction (shaded) and expansion phases are also shown together with the piecewise linear trend (red) and the individual phase growth rates.

Table 2. New Zealand classical business cycle phases 1954q2 – 2025q3. Duration D, amplitude A and growth rate $\hat{\mu} = A/D$ are given for each contraction and expansion phase. Peaks and troughs reflect Bry and Boschan (1971) dating of updated Hall and McDermott (2011, 2016) real New Zealand GDP.

	Contraction				Expansion			
Peak	D (qrs)	A (%)	$\hat{\mu}$ (%)	Trough	D (qrs)	A (%)	$\hat{\mu}$ (%)	Peak
1966q4	4	-2.50	-0.63	1967q4	34	31.78	0.93	1976q2
1976q2	7	-4.20	-0.60	1978q1	17	10.19	0.60	1982q2
1982q2	3	-3.25	-1.08	1983q1	19	13.56	0.71	1987q4
1987q4	4	-1.49	-0.37	1988q4	8	2.39	0.30	1990q4
1990q4	2	-3.13	-1.56	1991q2	24	23.54	0.98	1997q2
1997q2	3	-0.94	-0.31	1998q1	39	35.51	0.91	2007q4
2007q4	6	-2.55	-0.43	2009q2	42	30.40	0.72	2019q4
2019q4	2	-12.03	-6.02	2020q2	15	20.46	1.36	2024q1
2024q1	2	-1.87	-0.94	2024q3				
Median	3.00	-2.55	-0.63		21.50	22.00	0.82	
Mean	3.67	-3.55	-1.33		24.75	20.98	0.82	
IQR	2.00	1.37	0.66		18.75	18.03	0.26	
Std Dev	1.80	3.33	1.80		12.28	11.59	0.31	

recession, 2007q4 to 2009q2. The standout amplitude has been a decline of 12.0 percent for the Covid-19 recession, commensurate with a quarterly growth rate decline of 6.0 percent.

For New Zealand's 8 completed expansion phases since the trough of 1967q4, the mean duration has been 24.8 quarters. The mean phase amplitude has been growth of 21.0 percent, with a modest mean growth rate per quarter of 0.8 percent. The two longest durations have been relatively recent: 42 quarters following the GFC recession and 39 quarters following the short 3 quarter Asian Financial Crisis (AFC) recession. The post-Covid-19 expansion was a modest 15 quarters. The phase amplitudes for the expansions following the GFC and AFC recessions have been amongst the highest, at 30.4 and 35.5 percent respectively, but their long durations have meant their quarterly growth rates were only 0.7 and 0.9 percent. In contrast, the amplitude of the 15 quarter expansion following the Covid-19 recession was a substantial 20.5 percent, but with around 70 percent of this occuring in the first quarter.

As a consequence, the post Covid-19 expansion has provided by far the greatest quarterly growth rate of 1.4 percent.

For contraction phases, the long-duration GFC recession did not coincide with the deepest amplitude and quarterly growth rate decline of the modest-duration Covid-19 recession and, from Table 2 alone, it is difficult to see commonality between the various phases in terms of their duration, amplitude and quarterly growth rates. There are somewhat closer associations between these three measures for the expansion phases, though none of the three expansions with longest durations and largest amplitudes provided the two largest quarterly growth rates which followed the Covid-19 and early 1990s recessions.

These observations suggest that measures of duration and amplitude alone may not capture key business cycle phase characteristics sufficiently well, and that other measures such as phase shape should additionally be determined, with all measures considered both singly and collectively.

Table 3 gives the turning points for each phase together with their total excess E_*, mean excess $\bar{E}_*$ and standardised excess $\tilde{E}_*$ over constant quarterly growth. Also shown are the values of total excess F (cumulated loss/gain) relative to no growth across the phase.

Results for the Harding and Pagan (2016) measure F show the GFC and Covid-19 recessions have been among the most severe, at 7.3 percent and 7.1 percent cumulated loss of GDP relative to no growth. The former loss is primarily associated with its lengthy 6 quarter duration, and the latter with its particularly large amplitude decline of 12.0 percent and its quarterly growth rate, a decline of 6.0 percent which is considerably greater than the mean growth rate per quarter, a decline of 1.3 percent. For expansions, the wide range of F values reflects the fact that the magnitude of F is typically dominated by $\frac{1}{2}DA$ (see (10) and the discussion that follows) whose variability is directly affected by the relatively large variability of expansion durations and amplitudes by comparison to contractions.

Note that, by virtue of their construction, all three excess measures E_*, $\bar{E}_*$ and $\tilde{E}_*$ always have the same sign, but with magnitudes that differ according to the the extent to which they are or are not standardised for duration and amplitude.

For both severities of recessions and strengths of expansions, the commentary given in the following sections is confined to results from the peak of 1987q4, as prior to that New Zealand's small open economy had considerably different policy regimes, and was subject to somewhat different types of domestic and international economic shocks.

Table 3. New Zealand classical business cycle phases 1954q2 – 2025q3. Total excess E_*, mean excess $\bar{E}_*$ and standardised excess $\tilde{E}_*$ over constant quarterly growth are given for each phase together with the total excess F over no growth.

	Contraction					Expansion				
Peak	F (%)	E_* (%)	$\bar{E}_*$ (%)	$\tilde{E}_*$ (%)	Trough	F (%)	E_* (%)	$\bar{E}_*$ (%)	$\tilde{E}_*$ (%)	Peak
1966q4	-4.32	0.68	0.17	6.84	1967q4	583.24	43.02	1.27	3.98	1976q2
1976q2	-12.77	1.92	0.27	6.54	1978q1	68.85	-17.79	-1.05	-10.27	1982q2
1982q2	-4.48	0.39	0.13	4.03	1983q1	178.45	49.67	2.61	19.28	1987q4
1987q4	-2.60	0.38	0.10	6.41	1988q4	7.45	-2.12	-0.27	-11.08	1990q4
1990q4	-4.01	-0.89	-0.44	-14.16	1991q2	259.00	-23.42	-0.98	-4.15	1997q2
1997q2	-1.11	0.30	0.10	10.64	1998q1	716.88	24.51	0.63	1.77	2007q4
2007q4	-7.32	0.33	0.06	2.17	2009q2	578.59	-59.74	-1.42	-4.68	2019q4
2019q4	-7.12	4.92	2.46	20.43	2020q2	245.55	92.10	6.14	30.01	2024q1
2024q1	-1.52	0.35	0.18	9.36	2024q3					
Median	-4.32	0.38	0.13	6.54		252.28	11.20	0.18	-1.19	
Mean	-5.03	0.93	0.34	5.80		329.75	13.28	0.87	3.11	
IQR	4.52	0.35	0.08	5.33		428.70	63.88	2.60	13.88	
Std Dev	3.62	1.65	0.82	9.13		262.67	48.44	2.53	14.55	

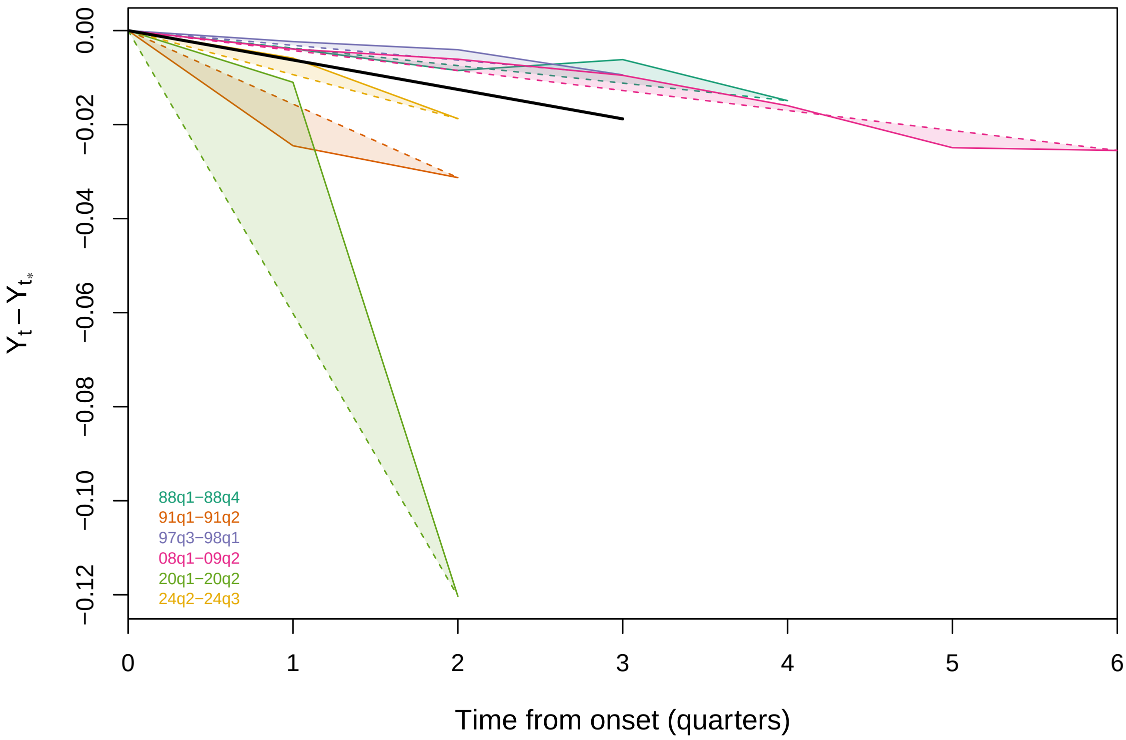

Figure 4. Phase plots of the paths $Y_{t_*+k} - Y_{t_*}$ (solid lines) of New Zealand $\log GDP$ contractions over the period 1987q4 to 2024q3 and their constant growth rate trends $\hat{Y}_{t_*+k} - Y_{t_*}$ (dashed lines) plotted against time from onset (quarters). The shaded areas depict E_* with areas above $\hat{Y}_{t_*+k} - Y_{t_*}$ counted as positive and the areas below as negative. A line denoting the median duration and median growth rate is superimposed (solid black).

Furthermore, to conserve space, we restrict attention to the total excess E_* and the standardised excess $\tilde{E}_*$ since these measures provide the more useful and interesting economic interpretations. In particular, plots of E_* provide an informative graphical display of the phase paths and key parameters D, A, $\hat{\mu}$ under study, and $\tilde{E}_*$ provides a measure that can be used to compare phase shape across phases of the same type (contraction or expansion), but different durations and amplitudes.

3.2 Severity of recessions

Figure 4 plots the phase paths $Y_{t_*+k} - Y_{t_*}$ of the 6 contraction phases in New Zealand $\log GDP$ over the period 1987q4 to 2024q3 and their constant growth rate trends $\hat{Y}_{t_*+k} - Y_{t_*}$, with both plotted against time from onset (quarters). Also shown are the individual phase durations D, amplitudes A, quarterly growth rates $\hat{\mu}$, together with E_*, the total excess over constant quarterly growth. A line corresponding to the median duration and median growth rate is superimposed for reference.

From Figure 4 we observe that the contraction phases fall into two groups each of three phases, one group with quarterly growth rates exceeding the median growth rate and the other below. Those above the median growth rate are broadly comparable in terms of quarterly growth rates and durations (all greater than the median duration), whereas those below the median growth rate are more anomalous, particularly the Covid-19 recession, despite all three having the same duration (2 quarters). In terms of total excess E_*, it is evident that the Covid-19 recession has by far the largest positive excess and the 2 quarter early 1990s recession has the most extreme negative excess. From Table 3 we confirm that, of the contractions considered, this is the only one with negative excess with the rest out-performing

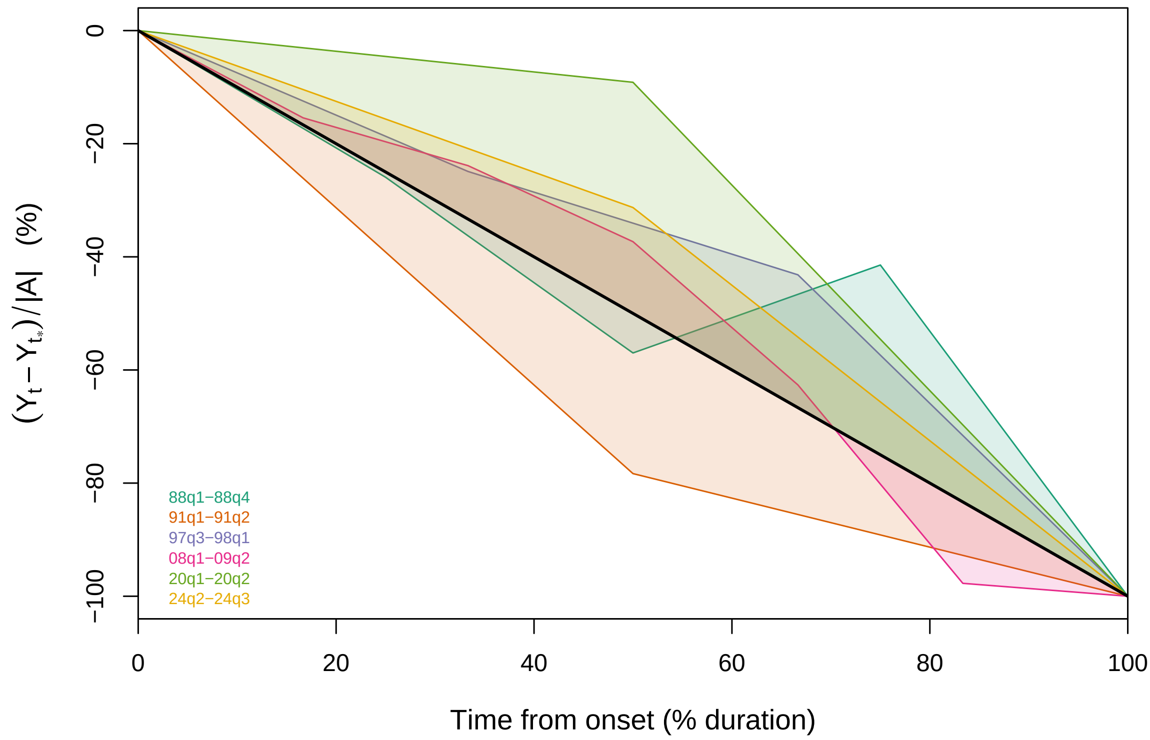

Figure 5. Phase plots of the standardised paths $(Y_{t_*+k} - Y_{t_*})/|A|$ of New Zealand $\log GDP$ contractions over the period 1987q4 to 2024q3 and their constant growth rate trends $(\hat{Y}_{t_*+k} - Y_{t_*})/|A|$ (solid black) plotted against time from onset (percent duration). The shaded areas depict $\tilde{E}_*$ with areas above $(\hat{Y}_{t_*+k} - Y_{t_*})/|A|$ counted as positive and the areas below as negative.

their quarterly growth rates.

To compare economic outcomes across contraction phases, we now consider Figure 5 which plots the standardised phase paths $(Y_{t_*+k} - Y_{t_*})/|A|$ of the New Zealand $\log GDP$ contraction phases over the period 1987q4 to 2024q3 together with $\tilde{E}_*$, the standardised excess over constant quarterly growth. The values of $\tilde{E}_*$ are given in Table 3.

The standardised contraction phase paths shown in Figure 5 are more comparable, as expected. Nevertheless, in order of absolute magnitude, the Covid-19 recession clearly has the greatest standardised excess (20.4 percent) and the early 1990s recession the least (-14.2 percent). The standardised excesses for the AFC and 2024 recessions are also sizable (10.6 and 9.4 percent). Note that the range of shapes of the contraction phase paths reflect recessions that are either consistently above, consistently below, or have periods above and below constant quarterly growth.

For the 6 recessions considered, all bar one, have out-performed their quarterly growth rates to varying degrees with the Covid-19 recession being the standout.

3.3 Strength of expansions

Figure 6 plots the phase paths $Y_{t_*+k} - Y_{t_*}$ of the 5 expansion phases in New Zealand $\log GDP$ over the period 1987q4 to 2024q3 and their constant growth rate trends $\hat{Y}_{t_*+k} - Y_{t_*}$, with both plotted against time from onset (quarters). Also shown are the individual phase durations D, amplitudes A, quarterly

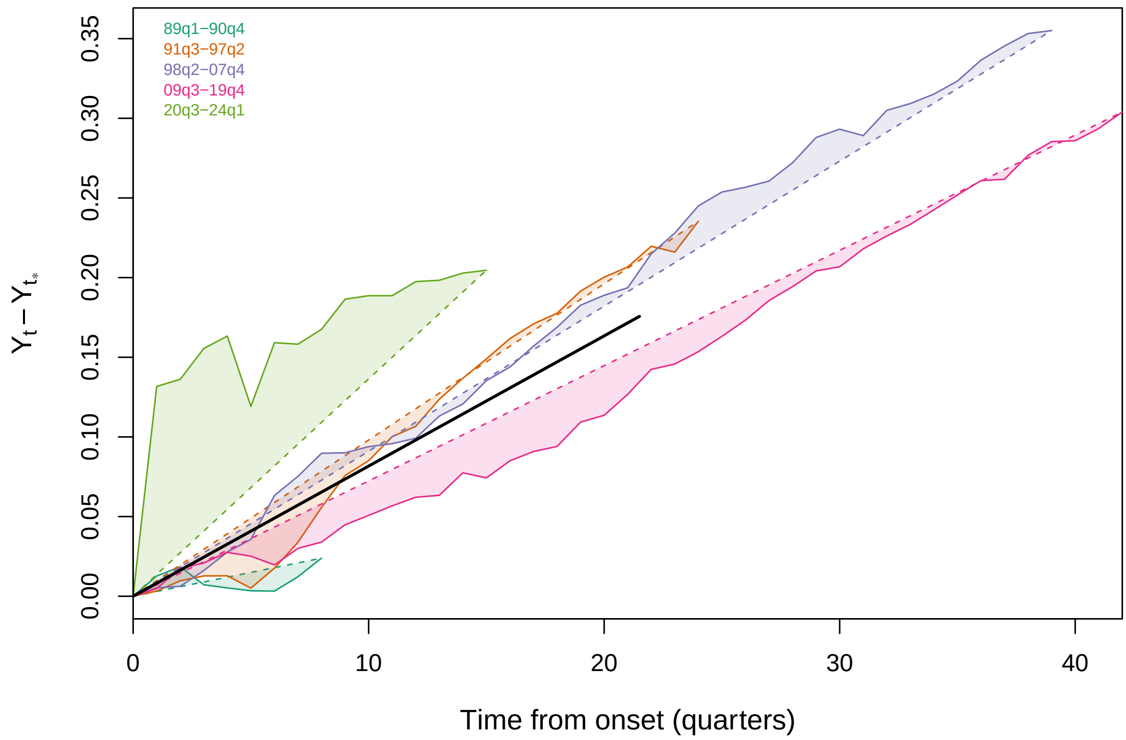

Figure 6. Phase plots of the paths $Y_{t_*+k} - Y_{t_*}$ (solid lines) of New Zealand $\log GDP$ expansions over the period 1987q4 to 2024q3 and their constant growth rate trends $\hat{Y}_{t_*+k} - Y_{t_*}$ (dashed lines) plotted against time from onset (quarters). The shaded areas depict E_* with areas above $\hat{Y}_{t_*+k} - Y_{t_*}$ counted as positive and the areas below as negative. The line denoting the median duration and median growth rate is superimposed (solid black).

growth rates $\hat{\mu}$, together with E_*, the total excess over constant quarterly growth. A line corresponding to the median duration and median growth rate is superimposed for reference.

As for contractions, Figure 6 shows that the expansion phases can be regarded as falling into two broad groups, one group with quarterly growth rates exceeding the median growth rate and the other below. Of those above the median growth rate, two are comparable in terms of quarterly growth rates and durations (all greater than the median duration), but the post Covid-19 expansion evidently has the greatest quarterly growth rate and total excess E_*. The two expansion phases below the median growth rate are more anomalous, with the post GFC expansion having the longest duration (42 quarters), but poor economic performance (smallest total excess E_*), whereas the late 1980s expansion has the lowest quarterly growth rate, shortest duration, and weak economic performance (negative total excess E_*). From Table 3 we note that, of the 5 expansion phases, only two had positive total excess E_* (the post Covid-19 and post AFC expansions) indicating that these phases had out-performed their quarterly growth rates.

Figure 7 plots the standardised phase paths $(Y_{t_*+k} - Y_{t_*})/|A|$ of the New Zealand $\log GDP$ contraction phases over the period 1987q4 to 2024q3 together with $\tilde{E}_*$, the standardised excess over constant quarterly growth. The values of $\tilde{E}_*$ (see Table 3) and Figure 7 can be used to compare economic outcomes across expansion phases.

The standardised expansion phase paths shown in Figure 7 are broadly comparable, with the exception of the post Covid-19 expansion which has by far the greatest standardised excess (30.0 percent) and the late 1980s expansion which has the smallest standardised excess (-11.1 percent). The latter

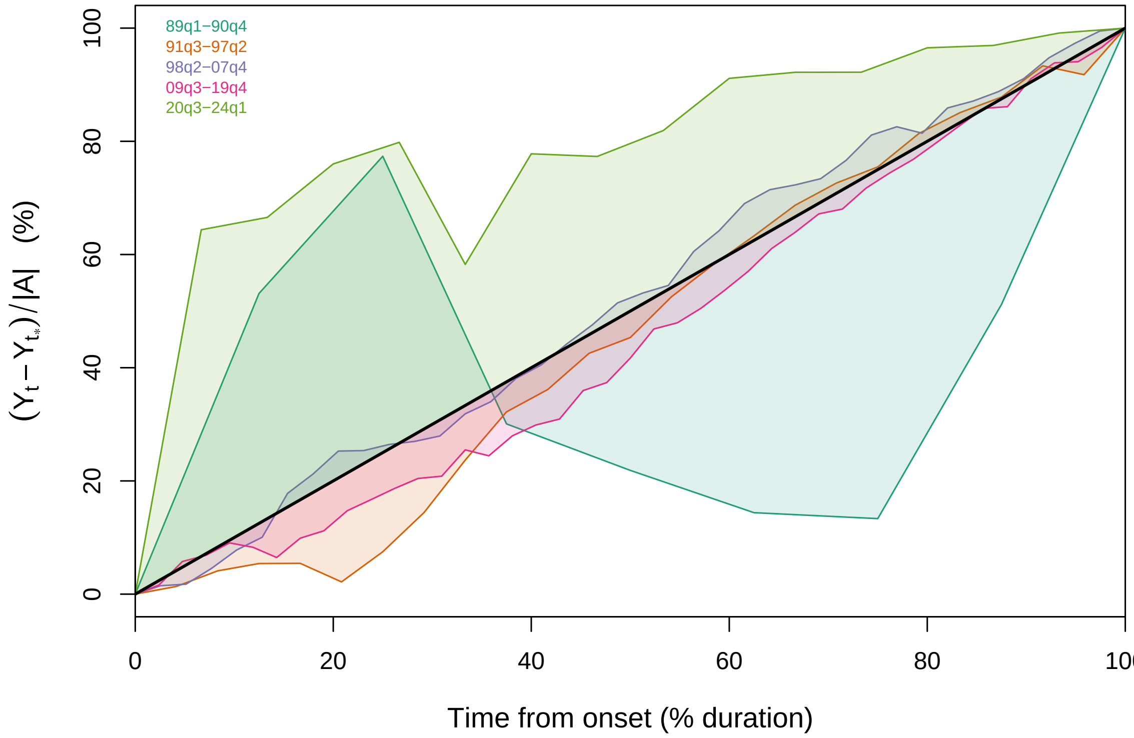

Figure 7. Phase plots of the standardised paths $(Y_{t_*+k} - Y_{t_*})/|A|$ of New Zealand $\log GDP$ expansions over the period 1987q4 to 2024q3 and their constant growth rate trends $(\hat{Y}_{t_*+k} - Y_{t_*})/|A|$ (solid black) plotted against time from onset (percentage duration). The shaded areas depict $\tilde{E}_*$ with areas above $(\hat{Y}_{t_*+k} - Y_{t_*})/|A|$ counted as positive and the areas below as negative.

presents a more complex standardised phase path, perhaps as a consequence of its very flat growth rate (0.3 percent) and short duration (8 quarters). Of the three remaining expansions, only the post AFC expansion has a positive standardised excess (1.8 percent) with the other two having negative standard excesses (-4.2 and -4.7 percent). The latter three expansion phases all have standardised excess values that are small in magnitude showing only modest departures from the benchmark constant quarterly growth.

For the 5 expansion phases considered, only two have yielded positive values for the standardised excess, namely the post Covid-19 expansion (30.0 percent) and the post AFC expansion (1.8 percent). Like the Covid-19 recession, the expansion that followed is a standout in terms of out-performing constant quarterly growth with the remaining expansions yielding much the same or worse economic outcomes than constant quarterly growth.

4 Conclusions

We develop three measures for the shape of business cycle phases, reflecting excess gains and losses in $\log GDP$ relative to constant quarterly growth across each phase. These relative measures represent better or worse economic outcomes during recessions and expansions, and provide summary evaluation measures for such economic outcomes. In essence, they measure the severity of recessions, strength of expansions and, as a consequence, can provide insights additional to those obtained from just duration and amplitude.

The starting point and benchmark for these measures is the piecewise linear trend of $\log GDP$ which joins the business cycle turning points by straight lines that have constant quarterly growth within phases. For each phase, the base measure E_* is just the total of the differences between $\log GDP$ and the piecewise linear trend so that it measures the total excess $\log GDP$ over constant quarterly growth. Normalising E_* by phase duration yields $\bar{E}_*$ which measures the mean excess $\log GDP$ over constant quarterly growth and scaling $\bar{E}_*$ by phase amplitude yields the standardised excess over constant quarterly growth. The standardised excess is dimensionless, bounded in magnitude by 0.5, and identical to the excess area measure proposed by Harding and Pagan (2016) apart from a constant of proportionality. The standardised excess $\tilde{E}_*$ can be used to compare phase shapes of the same type (contraction or expansion), but different durations and amplitudes. While all three measures share the same sign, they yield different values, different interpretations and, in many cases, different rankings. All three are useful for graphical as well as quantitative analysis, and have valuable economic interpretations.

Empirical results for New Zealand's contraction and expansion phases from 1954q2 to 2025q3 are presented with a detailed analysis of the three shape measures applied to a subset of this data (the 11 phases from 1987q4 to 2025q3.) For the 6 recessions considered, all bar one out-performed their quarterly growth rates to varying degrees with the Covid-19 recession being the standout. For the 5 expansions considered, only two gave positive values for the standardised excess, namely the post Covid-19 expansion (30.0 percent) and the post AFC expansion (1.8 percent). Like the Covid-19 recession, the expansion that followed is a standout in terms of out-performing constant quarterly growth with the remaining expansions yielding much the same or worse economic outcomes than constant quarterly growth.

Phase plots of the three excess measures have been developed to guide and inform economic analysis and commentary. These plots depend on the phase paths being sufficiently smooth so that they are relatively straightforward to visualise and compare. While this is likely to be the case (larger scale economies for example), it may not always be the case for more noisy phase paths or where we wish to compare a larger number of phases. In such cases the phase paths can be lightly smoothed to remove visually distracting noise at little cost to the accuracy of the excess measures.

Can these measures and their history over completed phases be used to assist in the analysis and prediction of incomplete current phases? How might we use these measures to investigate the implications for recovery paths (see discussion in Hall and McDermott, 2016)? These and other such questions are topics for further research.

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A Appendix

A.1 Smoothed path analysis

While the piecewise-linear trend $\hat{Y}_t$ generally provides an excellent description of the trajectory of Y_t as a whole, it does not directly depict the shape of Y_t within each phase. In particular, the shape of persistent deviations above or below the constant growth rate trend within each phase is not captured by $\hat{Y}_t$. In this section we develop other parsimonious trends that do convey shape information while satisfying the phase constraints that they pass through the turning points (t_j, Y_{t_j}). To satisfy the latter, our strategy is to model the trends in the growth rates across each phase using simple parsimonious models, and then integrate these to obtain suitable trends for Y_t.

Consider a typical phase with onset t_*, duration D, and model its growth rates as

$$\Delta Y_{t_*+k} = Y_{t_*+k} - Y_{t_*+k-1} = g_k + \epsilon_k \qquad (k = 1, \ldots, D) \tag{11}$$

where g_k denotes a simple trend and ϵ_k a zero-mean stationary residual, both local to the phase chosen. For simplicity, the dependence of these two components on the chosen phase has been suppressed. Some simple parsimonious models for g_k are

$$g_k = \mu \tag{12}$$
$$g_k = \alpha + \beta k \tag{13}$$
$$g_k = g_{k-1} + \eta_k \tag{14}$$

among many others. Here model (12) is the same as (3) and models constant growth rate across the phase whereas model (13) models linear growth across the phase. The random walk model (14) is a locally constant stochastic trend that leads to simple exponential smoothing (see Durbin and Koopman, 2001).

The fitted values $\hat{g}_k$ of g_k are now integrated to give a fitted trend $\tilde{Y}_t$ for Y_t. For each phase set $\tilde{Y}_{t_*} = Y_{t_*}$ and define

$$\tilde{Y}_{t_*+k} = Y_{t_*} + \sum_{s=1}^{k} \hat{g}_s \qquad (k = 1, \ldots, D) \tag{15}$$

where $\tilde{Y}_t$ must also satisfy the constraint $\tilde{Y}_{t_*+D} = Y_{t_*+D}$. However

$$\tilde{Y}_{t_*+D} = Y_{t_*+D} + \sum_{k=1}^{D} (\hat{g}_k - \Delta Y_{t_*+k})$$

so that $\tilde{Y}_{t_*+D}$ will only equal Y_{t_*+D} if

$$\frac{1}{D} \sum_{k=1}^{T} (\Delta Y_{t_*+k} - \hat{g}_k) = 0 \tag{16}$$

or the *mean trend deviation of the quarterly growth rates across each phase is identically zero*. Equivalently, across each phase, the mean of $\hat{g}_k$ must equal the mean quarterly growth rate. This simple, but important, condition will be satisfied by almost all models of interest and, in particular, the models for g_k given by (12), (13) and (14) as we now show.

For the *constant growth rate model* (12), the OLS estimator of μ for each phase is given by

$$\hat{\mu} = \frac{1}{D} \sum_{k=1}^{D} \Delta Y_{t_*+k}$$

and the fitted trend to the levels is given by

$$\tilde{Y}_{t_*+k} = Y_{t_*} + \hat{\mu} k \qquad (k = 1, \dots, D) \tag{17}$$

which is just the benchmark piecewise-linear trend $\hat{Y}_t$ given by (1) so that $\tilde{Y}_t = \hat{Y}_t$. Across each phase, the mean trend deviation for the levels Y_t is given by $\bar{E}_*$, the mean quarterly excess over constant quarterly growth given by (4) and (8), and condition (16) is satisfied by virtue of the definition of $\hat{\mu}$.

For the *linear growth rate model* (13), the OLS estimators of α and β for each phase satisfy the normal equations

$$\sum_{k=1}^{D} (\Delta Y_{t_*+k} - \hat{\alpha} - \hat{\beta} k) = 0, \qquad \sum_{k=1}^{D} k(\Delta Y_{t_*+k} - \hat{\alpha} - \hat{\beta} k) = 0 \tag{18}$$

and the fitted trend to the levels is given by

$$\tilde{Y}_{t_*+k} = \hat{Y}_{t_*+k} - \tfrac{1}{2} \hat{\beta} k(D - k) \qquad (k = 1, \dots, D) \tag{19}$$

which gives a piecewise-quadratic trend for Y_t. Here

$$\hat{\beta} = \frac{12}{D^2 - 1} \frac{1}{D} \sum_{k=1}^{D} k(\Delta Y_{t_*+k} - \hat{\mu}) = -\frac{12}{D^2 - 1} \bar{E}_*$$

where the latter equation follows from (8). In this case condition (16) is satisfied and the mean trend deviation for the levels Y_t in each phase is given by

$$\frac{1}{D} \sum_{k=1}^{D} (Y_{t_*+k} - \tilde{Y}_{t_*+k}) = \frac{1}{D} \sum_{k=1}^{D} (D - k + 1)(\Delta Y_{t_*+k} - \hat{\alpha} - \hat{\beta} k) = 0$$

where these results follow from (18). The linear growth rate model has the distinction of producing trends in both the growth rates and the levels that pass through their respective data points with zero mean trend deviation. As a consequence

$$\sum_{k=1}^{D} (\tilde{Y}_{t_*+k} - \hat{Y}_{t_*+k}) = E_*$$

so the constant and linear growth rate models give the same values for E_*, $\bar{E}_*$ and $\tilde{E}_*$.

The *locally constant growth rate model* (14) can be fitted by minimising the criterion

$$\sum_{t=1}^{D} (\Delta Y_{t_*+k} - g_k)^2 + \lambda \sum_{k=2}^{D} (g_k - g_{k-1})^2$$

with respect to the g_k where $\lambda > 0$ is a tuning parameter. This yields the solution

$$\hat{\mathbf{g}} = (\mathbf{I} + \lambda \mathbf{G}' \mathbf{G})^{-1} \mathbf{x} \tag{20}$$

where $\hat{\mathbf{g}} = (\hat{g}_1, \dots, \hat{g}_D)'$, the $(D - 1) \times D$ matrix $\mathbf{G}$ has typical element $G_{ii} = 1$, $G_{ij} = -1$ $(j = i + 1)$, $\mathbf{x} = (\Delta Y_{t_*+1}, \dots, \Delta Y_{t_*+D})'$ and $\mathbf{I}$ is the D dimensional identity matrix. It is readily shown that the trend (20) satisfies condition (16) so that the fitted trend of the Y_{t_*+k} across each phase is now given by (15). In this case there is no guarantee that the mean trend deviation for the levels will be zero.

Like the HP filter, the tuning parameter λ of the locally constant growth model can be chosen by the analyst based on past experience or using other techniques (see Durbin and Koopman, 2001). As λ increases $\hat{g}_k$ becomes smoother, approaching $\hat{\mu}$ (constant growth rate model) in the limit. Conversely, as λ decreases to zero, $\hat{g}_k$ approaches the growth rates ΔY_{t_*+k} (no smoothing). As a consequence, the locally constant growth rate trend embodies a wide variety of flexible smooths that are likely to provide a more accurate picture of the underlying trend in a phase's growth rates and, in turn, the trend of the corresponding levels Y_{t_8+t}.

Note that the smoothed phase paths $\tilde{Y}_{t_*+k}$ can be used to construct less volatile estimates of the excess measures (4) and (7) based on the decomposition

$$E_* = \sum_{k=1}^{D}(Y_{t_*+k} - \tilde{Y}_{t_*+k}) + \sum_{k=1}^{D}(\tilde{Y}_{t_*+k} - \hat{Y}_{t_*+k}) \tag{21}$$

where the first summation represents accumulated noise and the second the systematic trend component. This decomposition leads to the *smoothed excess measures*

$$S_* = \sum_{k=1}^{D}(\tilde{Y}_{t_*+k} - \hat{Y}_{t_*+k}), \qquad \bar{S}_* = \frac{S_*}{D}, \qquad \tilde{S}_* = \frac{S_*}{D|A|} \tag{22}$$

where the fitted trend or smoothed path $\tilde{Y}_t$ can be determined from quarterly growth rate models such as those discussed. *As a consequence of the decomposition (21), the three smoothed excess measures S_*, $\bar{S}_*$, $\tilde{S}_*$ also have the same area interpretations as their counterparts E_*, $\bar{E}_*$, $\tilde{E}_*$.*

Examples of these measures are given in Table 4 which compares the excess measures E_* and $\tilde{E}_*$ with their smooth counterparts S_* and $\tilde{S}_*$ for the three expansion phases of New Zealand $\log GDP$ over the period 1990q4 to 2019q4. The constant (12), linear (13) and locally constant (14) growth rate models are considered. Note that the linear growth rate model (13) has been excluded from Table 4 since it gives the same excess measures as the constant growth rate model.

Table 4. Total excess E_* and standardised quarterly excess $\tilde{E}_*$ over constant quarterly growth for 3 expansion phases of New Zealand $\log GDP$ over the period 1990q4 to 2019q4 together with the corresponding smooth excess measures S_* and $\tilde{S}_*$ where the latter are based on the locally constant growth rate model with tuning parameter $\lambda = 2$. All figures are percentages.

Onset	E_*	S_*	$\tilde{E}_*$	$\tilde{S}_*$
1991q2	-23.42	-21.90	-4.15	-3.88
1998q1	24.51	24.43	1.77	1.76
2009q2	-59.74	-59.35	-4.68	-4.65

Figure 8 gives the corresponding expansion phase plots together with the fitted trend $\tilde{Y}_{t_*+k}$ for each of the quarterly growth rate trend models (12), (13) and (14). While the linear growth rate model provides a smooth approximation to the values of Y_t across the phase, it is not alway a good fit. In some cases (not shown) it can even give fitted values that fall outside the nominal range of the data (i.e. above the peak or below the trough for the phase). For these reasons it is unlikely to be favoured in practice. However, for suitably chosen λ (here $\lambda = 2$), the locally constant growth rate model (14) is both a smoother version of the data and, as well, can always be chosen so that the smoothed path $\tilde{Y}_t$ does satisfy boundary constraints. This flexibility is useful in practice and will typically provide simpler, smooth graphical summaries of phase shape and, in turn, a more stable estimate of the size of its excess.

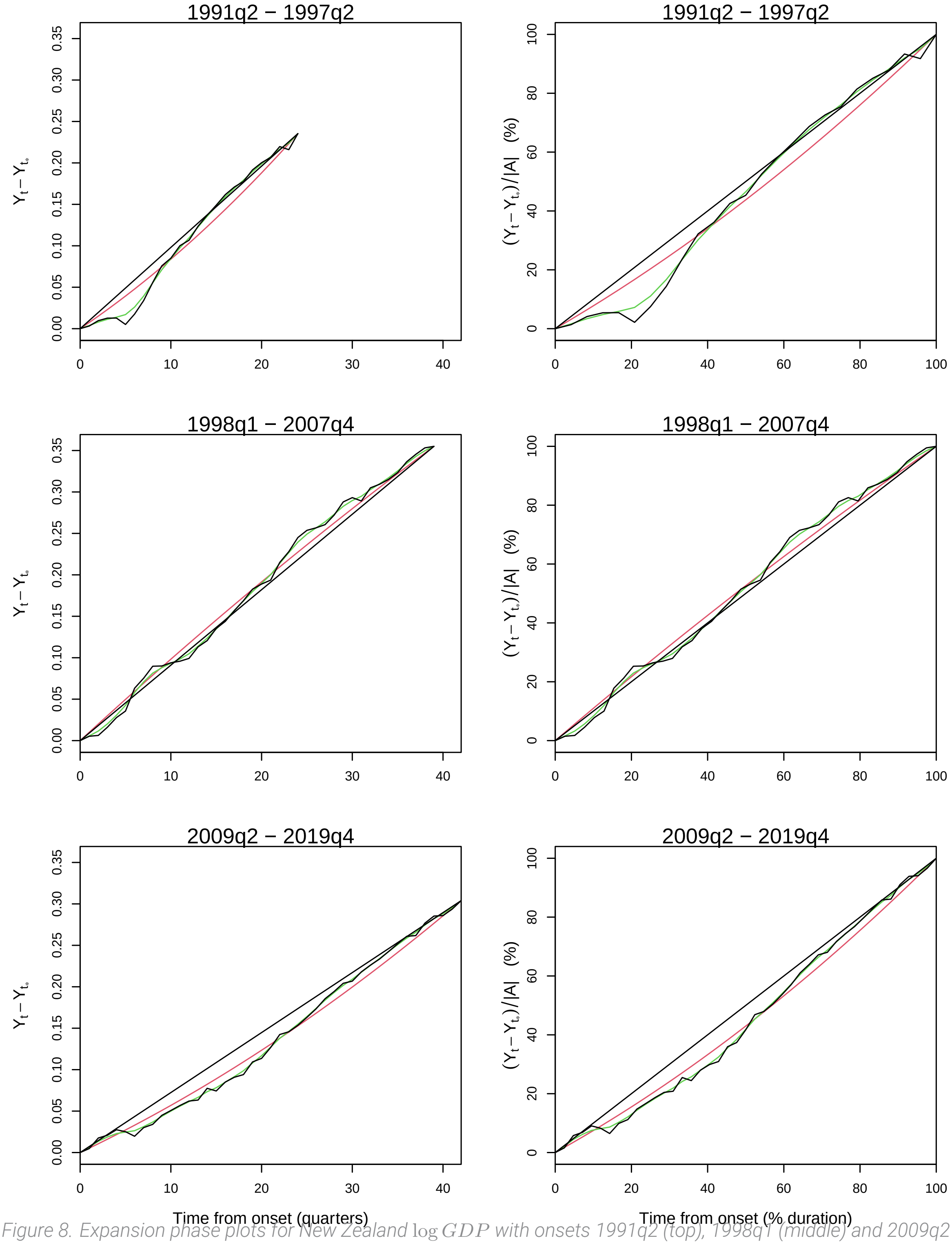

Figure 8. Expansion phase plots for New Zealand $\log GDP$ with onsets 1991q2 (top), 1998q1 (middle) and 2009q2 (bottom). The plots show the paths of $Y_{t_*+k} - Y_{t_*}$ (left, black) and their growth rate trends against time from onset (quarters), and the standardised paths $(Y_{t_*+k} - Y_{t_*})/|A|$ (right, black) and their trends against time from onset (percent duration). The constant (black), linear (red) and locally constant (green) growth rate trend models are superimposed.